approximate the exact ones to within $\sim\varepsilon$, $\varkappa$ in the interval of motion $T \sim \varepsilon^{-1}, \varkappa^{-1}$ respectively. In this sense the above controls are optimal in the first approximation.

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STABILITY OF THE PERIODIC SOLUTIONS OF QUASILINEAR AUTONOMOUS SYSTEMS WITH SEVERAL DEGREES OF FREEDOM

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Sufficient conditions for the asymptotic stability of quasilinear autonomous systems consisting of second-order equations are derived. The generating systems can have simple and multiple, commensurate and partly noncommensurate, and zero frequencies. The investigation is carried out with the aid of equations in variations for sufficiently small values of the parameter μ.

1. Let us consider the following quasilinear autonomous system with n degrees of freedom:

$$\sum_{k=1}^{n} (a_{ik}x_k^{\cdot\cdot} + c_{ik}x_k) = \mu F_i(x_1, \ldots, x_n, x_1^{\cdot}, \ldots, x_n^{\cdot}, \mu)$$

$$a_{ik} = a_{ki}, \qquad c_{ik} = c_{ki} \qquad (i = 1, \ldots, n)$$

We assume that the functions F_i are analytic in x_k and x_k within the ranges of these parameters, and also in the small parameter μ for $0 \leqslant \mu < \mu_0$.

Let the generating system (for $\mu = 0$) be a linear conservative system with constant coefficients whose kinetic energy is given by a homogeneous positive-definite quadratic form in the velocities and whose potential energy is a positive-definite form in the system coordinates.

The oscillation frequencies are given by the equation

$$\Delta(\omega^2) = |c_{ik} - \omega^2 a_{ik}| = 0 \tag{1.2}$$

Under the above conditions all of the roots of this equation are real and nonnegative. Some of them may be multiple roots.

Let us isolate some group of commensurate frequencies from the complete set of frequencies of the generating system. Let this group contain frequencies with subscripts from 1 to l and let

$$\omega_r = k_r \omega_0 \qquad (r = 1, \ldots, l) \tag{1.3}$$

where k_r is a positive integer. We also include in this group all zero frequencies with subscripts from $l+1$ to m,

$$\omega_r = 0 \qquad (r = l+1, \ldots, m) \tag{1.4}$$

The second group consists of all the remaining frequencies with subscripts from $m+1$ to n which are not commensurate with the frequencies with the subscripts $r = 1, \ldots, l$. We shall refer to these as the "noncommensurate" frequencies.

The frequency ω_0 is the frequency of the chosen single-frequency periodic solution of the generating system and $T_0 = 2\pi/\omega_0$ is the period of this solution. The period of the solution of quasilinear system (1.1) is a function of the small parameter μ.

The generating system can always be transformed into normal coordinates. The quasilinear system is then transformed into the corresponding quasinormal coordinates. A time transformation can be effected to make the period of the solution of the quasilinear system independent of the parameter μ and equal to T_0. As a result of these transformations system (1.1) becomes

$$z_r'' + \omega_r^2 z_r = \mu \Phi_r(z_1, \ldots, z_n, z_1', \ldots, z_n', \mu) \quad (r = 1, \ldots, n) \tag{1.5}$$

Let us take $z_1'(0) = 0$. as one of our initial conditions. The formulas for the functions $z_r(\tau)$ are [1, 2]

$$z_r(\tau) = (A_{r0} + \beta_r)\cos\omega_r\tau + \frac{B_{r0} + \gamma_r}{\omega_r}\sin\omega_r\tau + \sum_{k=1}^{\infty}\left[C_{rk}(\tau) + \cdots\right]\mu^k$$

$$B_{10} = 0, \quad \gamma_1 = 0 \qquad (r = 1, \ldots, l) \tag{1.6}$$

The quantities β_r and γ_r are equal to zero for $\mu = 0$. For $r = l+1, \ldots, m$ the functions $z_r(\tau)$ can be obtained from formula (1.6) by taking its limit as $\omega_r \to 0$. For $r = m+1, \ldots, n$ the quantities A_{r0} and B_{r0} are equal to zero.

The functions $C_{rk}(\tau)$ are given by the formulas

$$C_{rk}(\tau) = \frac{1}{\omega_r}\int_0^\tau H_{rk}(\tau_1)\sin\omega_r(\tau - \tau_1)\,d\tau_1 \quad (r = 1, \ldots, l, m+1, \ldots, n)$$

$$C_{rk}(\tau) = \int_0^\tau H_{rk}(\tau_1)(\tau - \tau_1)\,d\tau_1 \qquad (r = l+1, \ldots, m) \tag{1.7}$$

From now on we shall need only the functions $C_{r1}(\tau)$ and their first derivatives with respect to τ for $\tau = T_0$. Here $H_{r1}(\tau) = \Phi_r(z_{s0}\, z_{s0}', 0)$.

The amplitudes A_{r0} and B_{r0} can be determined from the system of equations

$$C_{r1}(T_0) = 0 \quad (r = 1, \ldots, l), \quad C_{r1}'(T_0) = 0 \quad (r = 2, \ldots, m) \tag{1.8}$$

Let us assume that the amplitude equations have simple solutions, i. e. that the functional determinant of system (1. 8) is different from zero (the argument T_0 has been omitted),

$$\Delta^* = \frac{D(C_{11}, \ldots, C_{l1}, C_{21}', \ldots, C_{m1}')}{D(A_{10}, \ldots, A_{m0}, B_{20}, \ldots, B_{l0})} \neq 0 \tag{1.9}$$

In this case all the functions $z_r(\tau)$ can be expanded in whole powers of the parameter μ.

We note that the initial condition $z_1'(0) = 0$ implies that $C_{11}'(T_0) = 0$. Since this equation is satisfied identically, all of the derivatives of $C_{11}'(T_0)$ with respect to A_{r0} and B_{r0} are also equal to zero.

2. Let us construct the equations in variations of system (1.5),

$$y_r'' + \omega_r^2 y_r = \mu \sum_{s=1}^{n} \left(\frac{\partial \Phi_r}{\partial z_s} y_s + \frac{\partial \Phi_r}{\partial z_s'} y_s' \right) \quad (r = 1, \ldots, n) \tag{2.1}$$

Here the T_0-periodic solution under investigation has been substituted into the functions Φ_r.

We know [3] that each root of the characteristic equation of system (2. 1) is associated with at least one particular solution of this system of the form

$$y_r(\tau) = e^{\alpha_p \tau} u_r(\tau) \quad (r, p = 1, \ldots, n) \tag{2.2}$$

Here α_p are the characteristic exponents which become the roots of the fundamental equation of the system ($\alpha_{p0} = \pm i\omega_p$) for $\mu = 0$; $u_r(\tau)$ are T_0-periodic functions of τ. Two values of α_p are associated with each subscript p. For each multiple root of the characteristic equation the number of independent solutions of the form (2. 2) is equal to the number of groups belonging to the given root. The remaining values of α_p can be obtained from linearly dependent solutions of the same form.

Let us replace the functions $y_r(\tau)$ in Eqs. (2. 1) by the expressions given by (2. 2). This yields

$$u_r'' + 2\alpha_p u_r' + (\alpha_p^2 + \omega_r^2) u_r = \mu \sum_{s=1}^{n} \left[\left(\frac{\partial \Phi_r}{\partial z_s} + \alpha_p \frac{\partial \Phi_r}{\partial z_s'} \right) u_s + \frac{\partial \Phi_r}{\partial z_s'} u_s' \right]$$

$$(r = 1, \ldots, n) \tag{2.3}$$

The solution of this equation for $\mu = 0$ can be obtained in the form

$$u_{r0}(\tau) = U_r e^{i\nu_r \tau} \tag{2.4}$$

Substituting this expression into the equation, we obtain the possible values of the quantity ν_r

$$\nu_{r1} = \mp(\omega_p - \omega_r), \qquad \nu_{r2} = \mp(\omega_p + \omega_r) \tag{2.5}$$

Hence, if the subscripts r and p belong to the different groups of subscripts $(1, \ldots, m)$ and $(m + 1, \ldots, n)$, it follows that for $\mu = 0$ the T_0-periodic solution of Eq. (2. 3) can only be identically equal to zero (i. e. $u_{r0}(\tau) = 0$). This enables us to approximate the exponents α_p for the groups of commensurate and noncommensurate frequencies independently.

The elementary divisors of matrix (1. 2) are always simple under the above conditions. Hence, the number of any multiple root ω^2 of Eq. (1. 2) is always equal to the multiplicity of this root. The same applies to the frequencies ω themselves if they are not equal to zero. However, the zero frequencies combine in pairs to form two-frequency groups. This means that the elementary divisors of matrix (1.2) are second-degree divisors for the zero frequencies. The matrix of the fundamental equation of system (2.1) has the same elementary divisors as the matrix (1.2) with respect to the frequencies ω.

There have been several papers on the stability of quasilinear systems consisting of first-order equations [4–8].

In [8] it is shown that the characteristic exponents α_p and the functions $u_r(\tau)$ can be expanded in whole powers of $\mu^{1/\gamma}$, where γ is the multiplicity of the elementary divisors of the corresponding root of the fundamental equation of system (2.1). This requires the fulfillment of two additional conditions which will be formulated below.

Thus, for nonzero frequencies the quantities α_p and $u_r(\tau)$ must be sought in the form of series in whole powers of the parameter μ. If the index of the characteristic exponent α_p assumes values from 1 to l, then the constant part of the exponent $\alpha_{p0} = \pm i\omega_p$ can be omitted. This follows from the fact that the quantity $\exp(\alpha_{p0}\tau)$ is a T_0-periodic function of τ and can be included in the corresponding functions $u_r(\tau)$. We can therefore write

$$\alpha_p = \alpha_{p1}\mu + \ldots \quad (p = 1, \ldots, l), \qquad \alpha_p = \alpha_{p0} + \alpha_{p1}\mu + \cdots \qquad (2.6)$$

$$(p = m+1, \ldots, n) \qquad u_r(\tau) = u_{r0}(\tau) + \mu u_{r1}(\tau) + \ldots \quad (r = 1, \ldots, l, m+1, \ldots, n)$$

For the zero frequencies the expansions of the corresponding quantities must take the form of series in whole powers of $\mu^{1/2}$. Let us replace the functions $u_r(\tau)$ by the functions $v_r(\tau)$. Since $\alpha_{p0} = 0$ in this case, it follows that

$$\alpha_p = \alpha_{p_{1/2}}\mu^{1/2} + \alpha_{p1}\mu + \ldots \quad (r, p = l+1, \ldots, m)$$

$$v_r(\tau) = v_{r0}(\tau) + \mu^{1/2} v_{r_{1/2}}(\tau) + \mu v_{r1}(\tau) + \ldots \qquad (2.7)$$

From now on we shall compute the coefficients of the expansions of the exponents α_p up to and including the coefficient α_{p1}.

3. Let us find the coefficients α_{p1} of the characteristic exponents for $p = 1, \ldots, l$. Since in this case $\alpha_{p0} = 0$, we have

$$\begin{aligned} u_{r0}(\tau) &= P_r \cos\omega_r\tau + \frac{Q_r}{\omega_r}\sin\omega_r\tau \quad (r = 1, \ldots, l) \\ u_{r0}(\tau) &= P_r \quad (r = l+1, \ldots, m) \end{aligned} \qquad (3.1)$$

Let us construct the equation for the functions $u_{r1}(\tau)$ for $r = 1, \ldots, l$

$$u_{r1}'' + \omega_r^2 u_{r1} = -2\alpha_{p1}u_{r0}' + \sum_{s=1}^{m}\left(\frac{\partial\Phi_r}{\partial z_s}\right)_0 u_{s0} + \sum_{s=1}^{l}\left(\frac{\partial\Phi_r}{\partial z_s'}\right)_0 u_{s0}' \qquad (3.2)$$

The subscript "0" appearing next to the derivatives of the functions Φ_r means z_s, z_s' and μ in these derivatives have been replaced by z_{s0}, z_{s0}' and 0.

Transforming the right side of Eq. (3.2), we obtain the self-evident relations

$$\begin{aligned} \left(\frac{\partial\Phi_r}{\partial A_{s0}}\right)_0 &= \left(\frac{\partial\Phi_r}{\partial z_s}\right)_0 \cos\omega_s\tau - \left(\frac{\partial\Phi_r}{\partial z_s'}\right)_0 \omega_s \sin\omega_s\tau \quad (s = 1, \ldots, m) \\ \left(\frac{\partial\Phi_r}{\partial B_{s0}}\right)_0 &= \frac{1}{\omega_s}\left[\left(\frac{\partial\Phi_r}{\partial z_s}\right)_0 \sin\omega_s\tau + \left(\frac{\partial\Phi_r}{\partial z_s'}\right)_0 \omega_s\cos\omega_s\tau\right] \quad (s = 2, \ldots, l) \end{aligned} \qquad (3.3)$$

These formulas readily yield yet another relation,

$$\frac{1}{A_{10}\omega_1^2}\left\{\sum_{s=2}^{l}\left[B_{s0}\left(\frac{\partial\Phi_r}{\partial A_{s0}}\right)_0 - A_{s0}\omega_s^2\left(\frac{\partial\Phi_r}{\partial B_{s0}}\right)_0\right] - \left(\frac{\partial\Phi_r}{\partial\tau}\right)_0\right\} =$$

$$= \left(\frac{\partial\Phi_r}{\partial z_1}\right)_0 \sin\omega_1\tau + \left(\frac{\partial\Phi_r}{\partial z_1'}\right)_0 \omega_1\cos\omega_1\tau \qquad (3.4)$$

Let us substitute the values of $u_{s0}(\tau)$ and $u_{s0}'(\tau)$ as given by formulas (3.1) into the right side of Eq. (3.2) and make use of the preceding relation. This gives us

$$u_{r1}'' + \omega_r^2 u_{r1} = -2\alpha_{p1}(P_r\omega_r\sin\omega_r\tau + Q_r\cos\omega_r\tau) + \sum_{s=1}^{m} P_s\left(\frac{\partial\Phi_r}{\partial A_{s0}}\right)_0 +$$

$$+\sum_{s=2}^{l} Q_s\left(\frac{\partial\Phi_r}{\partial B_{s0}}\right)_0 + \frac{Q_1}{A_{10}\omega_1^2}\left\{\sum_{s=2}^{l}\left[B_{s0}\left(\frac{\partial\Phi_r}{\partial A_{s0}}\right)_0 - A_{s0}\omega_s^2\left(\frac{\partial\Phi_r}{\partial B_{s0}}\right)_0\right] - \left(\frac{\partial\Phi_r}{\partial\tau}\right)_0\right\}$$

$$(r = 1, \ldots, l) \qquad (3.5)$$

Let us construct the conditions of periodicity of the functions $u_{r1}(\tau)$. We can do this by multiplying the right sides of Eq. (3.5) by $\sin\omega_r\tau/\omega_r$ and $\cos\omega_r\tau$ successively, integrating from zero to T_0, and equating the results to zero. Formulas (1.8) must be used in carrying out these operations.

Since all of the derivatives of $C_{11}'(T_0)$ with respect to A_{s0} and B_{s0} are equal to zero, it follows that one of the conditions of periodicity for $r = 1$ is of the form $Q_1\alpha_{p1}T_0 = 0$. Hence, if $Q_1 \neq 0$, then

$$\alpha_{p1} = 0 \qquad (3.6)$$

This coefficient corresponds to the zero characteristic exponent which autonomous systems always possess.

For $Q_1 = 0$ the remaining conditions of periodicity are

$$\begin{aligned}
&\sum_{s=1}^{m} P_s\left[\frac{\partial C_{r1}(T_0)}{\partial A_{s0}} - \delta_{sr}\alpha_{p1}T_0\right] + \sum_{s=2}^{l} Q_s\frac{\partial C_{r1}(T_0)}{\partial B_{s0}} = 0 \qquad (r = 1, \ldots, l)\\
&\sum_{s=1}^{m} P_s\frac{\partial C_{r1}'(T_0)}{\partial A_{s0}} + \sum_{s=2}^{l} Q_s\left[\frac{\partial C_{r1}'(T_0)}{\partial B_{s0}} - \delta_{sr}\alpha_{p1}T_0\right] = 0 \qquad (r = 2, \ldots, m)
\end{aligned} \qquad (3.7)$$

Here δ_{sr} is the Kronecker delta; $\delta_{sr} = 0$ for $s \neq r$, and $\delta_{rr} = 1$.

System (3.7) is a linear homogeneous system of $l + m - 1$ equation in P_s $(s = 1, \ldots, m)$ and Q_s $(s = 2, \ldots, l)$. The solution of these equations is nontrivial only if the determinant of system (3.7) is equal to zero, i.e. only if

$$\Delta_1 = \begin{vmatrix}
\frac{\partial M_{11}}{\partial A_{10}} & \ldots & \frac{\partial C_{11}}{\partial A_{l0}} & \frac{\partial C_{11}}{\partial B_{20}} & \ldots & \frac{\partial C_{11}}{\partial B_{l0}} & \frac{\partial C_{11}}{\partial A_{l+1,0}} & \ldots & \frac{\partial C_{11}}{\partial A_{m0}}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\frac{\partial C_{l1}}{\partial A_{10}} & \ldots & \frac{\partial M_{l1}}{\partial A_{l0}} & \frac{\partial C_{l1}}{\partial B_{20}} & \ldots & \frac{\partial C_{l1}}{\partial B_{l0}} & \frac{\partial C_{l1}}{\partial A_{l+1,0}} & \ldots & \frac{\partial C_{l1}}{\partial A_{m0}}\\
\frac{\partial C_{21}'}{\partial A_{10}} & \ldots & \frac{\partial C_{21}'}{\partial A_{l0}} & \frac{\partial M_{21}'}{\partial B_{20}} & \ldots & \frac{\partial C_{21}'}{\partial B_{l0}} & \frac{\partial C_{21}'}{\partial A_{l+1,0}} & \ldots & \frac{\partial C_{21}'}{\partial A_{m0}}\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{vmatrix} = 0 \qquad (3.8)$$

(cont. on next page)

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial C_{l1}'}{\partial A_{10}} & \cdots & \frac{\partial C_{l1}'}{\partial A_{l0}} & \frac{\partial C_{l1}'}{\partial B_{20}} & \cdots & \frac{\partial M_{l1}'}{\partial B_{l0}} & \frac{\partial C_{l1}'}{\partial A_{l+1,0}} & \cdots & \frac{\partial C_{l1}'}{\partial A_{m0}} \\ \frac{\partial C_{l+1,1}'}{\partial A_{10}} & \cdots & \frac{\partial C_{l+1,1}'}{\partial A_{l0}} & \frac{\partial C_{l+1,1}'}{\partial B_{20}} & \cdots & \frac{\partial C_{l+1,1}'}{\partial B_{l0}} & \frac{\partial C_{l+1,1}'}{\partial A_{l+1,0}} & \cdots & \frac{\partial C_{l+1,1}'}{\partial A_{m0}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial C_{m1}'}{\partial A_{10}} & \cdots & \frac{\partial C_{m1}'}{\partial A_{l0}} & \frac{\partial C_{m1}'}{\partial B_{20}} & \cdots & \frac{\partial C_{m1}'}{\partial B_{l0}} & \frac{\partial C_{m1}'}{\partial A_{l+1,0}} & \cdots & \frac{\partial C_{m1}'}{\partial A_{m0}} \end{vmatrix}$$

The order of the rows and columns in the determinant Δ_1 has been altered somewhat. The argument T_0 of the quantities $C_{r1}(T_0)$ and $C_{r1}'(T_0)$ has been omitted. Moreover, the notation

$$\frac{\partial M_{r1}}{\partial A_{r0}} = \frac{\partial C_{r1}}{\partial A_{r0}} - \alpha_{p1}T_0, \qquad \frac{\partial M_{r1}'}{\partial B_{r0}} = \frac{\partial C_{r1}'}{\partial B_{r0}} - \alpha_{p1}T_0 \tag{3.9}$$

has been introduced for brevity.

Expanding this determinant, we obtain an equation of degree $2l - 1$ in $\alpha_{p1}(p = 1, \ldots, l)$ from which we can find the remaining $2l - 1$ values (in addition to the zero value) of this coefficient of the characteristic exponents. The absolute term in equation (3.8) coincides with functional determinant (1.9) of the amplitude equations, which is not equal to zero by hypothesis. Thus, only one of the coefficients α_{p1} $(p = 1, \ldots, l)$ is equal to zero.

One of the two additional conditions [8] of expandability of the characteristic exponents in whole powers of the parameter μ for $p = 1, \ldots, l$ in this case consists in the linear independence of the last rows (or columns) of determinant (3.8) not containing the coefficient α_{p1}. This condition is fulfilled by virtue of (1.9).

The second condition requires that all the roots of the determinant equation $\Delta_1 = 0$ for the first coefficient α_{p1} be simple. Let us assume that this condition is fulfilled.

4. Now let us determine the characteristic exponents for $p = l + 1, \ldots, m$. Here we make use of expansions (2.7) for α_p and $v_r(\tau)$.

The functions $v_{r0}(\tau)$ are

$$v_{r0}(\tau) = R_{r0}\cos\omega_r\tau + \frac{S_{r0}}{\omega_r}\sin\omega_r\tau \qquad (r = 1, \ldots, l)$$
$$v_{r0}(\tau) = R_{r0} \quad (r = l+1, \ldots, m) \tag{4.1}$$

We have the following equations for the functions $v_{r^1/_2}(\tau)$:

$$v_{r^1/_2}'' + \omega_r^2 v_{r^1/_2} = -2\alpha_{p^1/_2}v_{r0}' \quad (r = 1, \ldots, l), \qquad v_{r^1/_2}'' = 0 \quad (r = l+1, \ldots, m)$$

Let us suppose that $\alpha_{p^1/_2} \neq 0$. The conditions of periodicity of the functions $v_{r^1/_2}(\tau)$ for $r = 1, \ldots, l$ then imply that

$$v_{r0}(\tau) = 0 \qquad (r = 1, \ldots, l) \tag{4.2}$$

Thus,

$$v_{r^1/_2}(\tau) = R_{r^1/_2}\cos\omega_r\tau + \frac{S_{r^1/_2}}{\omega_r}\sin\omega_r\tau \qquad (r = 1, \ldots, l)$$
$$v_{r^1/_2}(\tau) = R_{r^1/_2} \qquad (r = l+1, \ldots, m) \tag{4.3}$$

Let us construct equations for the functions $v_{r1}(\tau)$. We have

$$v_{r1}'' + \omega_r^2 v_{r1} = -2\alpha_{p^{1/2}} v_{r^{1/2}}' + \sum_{s=l+1}^{m} \left(\frac{\partial \Phi_r}{\partial z_s}\right)_0 v_{s0} \qquad (r = 1, \ldots, l) \tag{4.4}$$

$$v_{r1}'' = -\alpha_{p^{1/2}}^2 v_{r0} + \sum_{s=l+1}^{m} \left(\frac{\partial \Phi_r}{\partial z_s}\right)_0 v_{s0} \qquad (r = l+1, \ldots, m) \tag{4.5}$$

The conditions of periodicity of the functions $v_{r1}'(\tau)$ for $r = l+1, \ldots, m$ yield

$$\sum_{s=l+1}^{m} R_{s0} \frac{\partial C_{r1}'}{\partial A_{s0}} - \alpha_{p^{1/2}}^2 T_0 R_{r0} = 0 \qquad (r = l+1, \ldots, m) \tag{4.6}$$

This system of linear homogeneous equations in R_{s0} has a nontrivial solution if

$$\Delta_2 = \begin{vmatrix} \dfrac{\partial C_{l+1,1}'}{\partial A_{l+1,0}} - \alpha_{p^{1/2}}^2 T_0 & \cdots & \dfrac{\partial C_{l+1,1}'}{\partial A_{m0}} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial C_{m1}'}{\partial A_{l+1,0}} & \cdots & \dfrac{\partial C_{m1}'}{\partial A_{m0}} - \alpha_{p^{1/2}}^2 T_0 \end{vmatrix} = 0 \tag{4.7}$$

Expanding the determinant, we obtain an equation of degree $m - l$ in $\alpha_{p^{1/2}}^2$ for $p = l+1, \ldots, m$. Let us assume that all the roots of this equation are negative. Then all the coefficients $\alpha_{p^{1/2}}$ are purely imaginary.

In this case the first condition of expandability of the exponent in whole powers of $\mu^{1/2}$ is fulfilled automatically. We shall assume that the second condition as regards the nonmultiplicity of the roots of the determinant equation $\Delta_2 = 0$ has been satisfied.

Let us establish a correspondence between each value of the subscript "r" and specific values of the subscript "p" (e.g. by setting $p = r$). System (4.6) then enables us to determine the ratios of the quantities R_{s0} to any one of them, e.g. $R_{s0}^* = R_{s0}/R_{m0}$ for each value of $\alpha_{p^{1/2}}^2$. If all the roots of Eq. (4.7) are simple, then each root is associated with its own system of values R_{s0}^* $(s = l+1, \ldots, m)$.

The conditions of periodicity of the functions $v_{r1}(\tau)$ for $r = 1, \ldots, l$ are

$$\alpha_{p^{1/2}} T_0 R_{r^{1/2}} = \sum_{s=l+1}^{m} R_{s0} \frac{\partial C_{r1}}{\partial A_{s0}}, \qquad \alpha_{p^{1/2}} T_0 S_{r^{1/2}} = \sum_{s=l+1}^{m} R_{s0} \frac{\partial C_{r1}'}{\partial A_{s0}} \qquad (r = 1, \ldots, l) \tag{4.8}$$

The second formula of (4.8) implies that $S_{1^{1/2}} = 0$. Finally, let us construct the equations for the functions $v_{r^{3/2}}(\tau)$ for $r = l+1, \ldots, m$. We have

$$\begin{aligned} v_{r^{3/2}}'' = {} & -2\alpha_{p^{1/2}} v_{r1}' - \alpha_{p^{1/2}}^2 v_{r^{1/2}} - 2\alpha_{p^{1/2}} \alpha_{p1} v_{r0} + \sum_{s=1}^{m} \left(\frac{\partial \Phi_r}{\partial z_s}\right)_0 v_{s^{1/2}} + \\ & + \sum_{s=1}^{l} \left(\frac{\partial \Phi_r}{\partial z_s'}\right)_0 v_{s^{1/2}}' + \alpha_{p^{1/2}} \sum_{s=l+1}^{m} \left(\frac{\partial \Phi_r}{\partial z_s'}\right)_0 v_{s0} \qquad (r = l+1, \ldots, m) \end{aligned} \tag{4.9}$$

Let us note the following fact. If we replace the quantities R_{r0} by $R_{r^{1/2}}$ in Eqs. (4.6), then all our conclusions concerning Eqs. (4.6) remain valid for the new equations. Thus, each sum of terms of the form

$$\sum_{s=l+1}^{m} R_{s^{1/2}} \frac{\partial C_{r1}'}{\partial A_{s0}} - \alpha_{p^{1/2}}^2 T_0 R_{r^{1/2}} \qquad (r = l+1, \ldots, m)$$

can be made to vanish by suitable choice of the ratios $R_{s^1/_2}/R_{m^1/_2}$ if $\alpha_{p^1/_2}{}^2$ is one of the roots of Eq. (4.7).

This remark, formulas (3.3) and (4.8), and the condition of periodicity of the function $v_{r^1/_2}{}'(\tau)$ for $r = l+1, \ldots, m$ imply that

$$2\alpha_{p1}T_0R_{r0}^* = \qquad (r, p = l+1, \ldots, m) \tag{4.10}$$

$$= \sum_{s=l+1}^{m} R_{s0}^* \left[\int_0^{T_0} \left(\frac{\partial \Phi_r}{\partial z_s{}'}\right)_0 d\tau + \frac{1}{\alpha_{p^1/_2}^2 T_0}\left(\sum_{k=1}^{l} \frac{\partial C_{k1}}{\partial A_{s0}} \frac{\partial C_{r1}{}'}{\partial A_{k0}} + \sum_{k=2}^{l} \frac{\partial C_{k1}'}{\partial A_{s0}} \frac{\partial C_{r1}{}'}{\partial B_{k0}}\right)\right]$$

Formula (4.10) defines $2(m-l)$ equal pairs of values of the real coefficient α_{p1} for $p = l+1, \ldots, m$.

5. Finally let us consider the characteristic exponents for $p = m+1, \ldots, n$ corresponding to the noncommensurate frequencies. In this case the coefficients $\alpha_{p0} = \pm i\omega_p$. Let us break up the group of noncommensurate frequencies into subgroups in order to compute the coefficients α_{p1}. Each subgroup consists of some frequency ω_r and all the frequencies equal to it, as well as the frequencies satisfying one of the relations

$$|\omega_r - \omega_s| = c_s\omega_0, \quad \omega_r + \omega_s = c_s\omega_0 \tag{5.1}$$

where c_s is a positive integer. Thus, each of the noncommensurate frequencies occurs in only one of the subgroups. The number of subgroups can range from one to $n-m$. Three cases are possible for each subgroup.

a) The subgroup contains the frequency ω_r only. Then all the $u_{r0}(\tau) = 0$ for $r \neq p$ and we have $u_{r0}(\tau) = U_r$ only for $r = p$. In this case the condition of periodicity of the function $u_{r1}(\tau)$ for $r = p$ yields two complex conjugate values of the coefficient α_{r1}[9],

$$2\alpha_{r1}T_0 = \int_0^{T_0} \left[\left(\frac{\partial \Phi_r}{\partial z_r{}'}\right)_0 \mp \frac{i}{\omega_r}\left(\frac{\partial \Phi_r}{\partial z_r}\right)_0\right] d\tau \tag{5.2}$$

b) The subgroup contains only a multiple frequency ω_r of multiplicity j. We have the following determinant equation for determining the coefficients α_{p1}:

$$\left|\int_0^{T_0} \left[\left(\frac{\partial \Phi_q}{\partial z_s{}'}\right)_0 \mp \frac{i}{\omega_r}\left(\frac{\partial \Phi_q}{\partial z_s}\right)_0\right] d\tau - 2\delta_{qs}\alpha_{p1}T_0\right| = 0$$

$$(p, q, s = r, \ldots, r+j-1) \tag{5.3}$$

Equation (5.3) yields $2j$ complex conjugate values of the coefficient α_{p1}.

c) Let us consider the general case where in addition to the simple or multiple frequency ω_r the subgroup also contains frequencies satisfying relations (5.1). The functions $u_{r0}(\tau)$ for these frequencies are given by formulas (2.4) and (2.5). Let us construct the equation for the functions $u_{r1}(\tau)$,

$$u_{r1}'' + 2\alpha_{p0}u_{r1}' + (\alpha_{p0}^2 + \omega_r^2)u_{r1} = -2\alpha_{p1}(\alpha_{p0} + i\nu_r)U_re^{i\nu_r\tau} +$$

$$+ \sum_{s=m+k_i}^{m+k_j} \left\{\left[\left(\frac{\partial \Phi_r}{\partial z_s}\right)_0 + (\alpha_{p0} + i\nu_s)\left(\frac{\partial \Phi_r}{\partial z_s{}'}\right)_0\right] U_se^{i\nu_s\tau}\right\} \tag{5.4}$$

The subscript "s" in the latter sum assumes all the values belonging to the subgroup under consideration. The left side of Eq. (5.4) is of the same form as the left side of the equation for $u_{r0}(\tau)$. We therefore obtain the condition of periodicity of the function $u_{r1}(\tau)$ by multiplying the right side of Eq. (5.4) by $\exp(-i\nu_r\tau)$, integrating from 0 to T_0, and equating the result to zero. We have

$$\sum_{s=m+k_i}^{m+k_j}\left\{\int_0^{T_0}\left[\left(\frac{\partial\Phi_r}{\partial z_s}\right)_0+(\alpha_{p0}+i\nu_s)\left(\frac{\partial\Phi_r}{\partial z_s'}\right)_0\right]e^{i(\nu_s-\nu_r)\tau}d\tau\cdot U_s\right\}-$$

$$-2\alpha_{p1}(\alpha_{p0}+i\nu_r)T_0U_r=0 \qquad (5.5)$$

We substitute in the value $\nu_r=\nu_{r1}$ from formula (2.5) for the fundamental frequency ω_r and the frequencies related to it by the first relation of (5.1), and the value $\nu_r=\nu_{r2}$ for the frequencies related to the fundamental frequency by the second relation of(5.1). This yields the following equation for determining the coefficient α_{p1} :

$$\left|\int_0^{T_0}\left[\left(\frac{\partial\Phi_r}{\partial z_s}\right)_0\pm\alpha_{s0}\left(\frac{\partial\Phi_r}{\partial z_s'}\right)_0\right]e^{(\pm\alpha_{s0}\mp\alpha_{r0})\tau}d\tau\mp 2\delta_{sr}\alpha_{p1}\alpha_{r0}T_0\right|=0$$

$$(p,r,s=m+k_i,\ldots,m+k_j) \qquad (5.6)$$

The upper signs in these expressions apply to the frequencies related to the fundamental frequency by the first relation of (5.1); the lower signs apply to the frequencies related to it by the second relation of (5.1). Equation (5.6) yields $2(k_j-k_i)$ complex conjugate values of the coefficient α_{p1}. As before, we assume that the roots of Eq.(5.6) are simple.

Conclusion. We can draw the following conclusions for sufficiently small values of the parameter μ. The sufficient conditions of asymptotic stability of the periodic solutions of quasilinear autonomous (1.1), or of equivalent system (1.5) under the assumptions of Sect. 1 concerning the frequencies of the generating system are:(1) the imaginary character of all the coefficients $\alpha_{p^{1/2}}$ of the characteristic exponents for $p=l+$ $+1,\ldots,m$; (2) the negativeness of the real parts of all the coefficients α_{p1} for $p=1,\ldots,n$ with the exception of one coefficient which is equal to zero.

Throughout our discussion we assumed that the determinant equations for α_{p1} $(p=1,\ldots,l,\ m+1,\ldots,n)$ and for $\alpha_{p^{1/2}}$ $(p=l+1,\ldots,m)$ had simple roots. If these equations have multiple roots, then the characteristic equations can be expanded in other powers of the parameter μ than those assumed.

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STABILITY OF PERIODIC SOLUTIONS OF QUASILINEAR ELASTIC GYROSCOPIC SYSTEMS WITH DISTRIBUTED AND CONCENTRATED PARAMETERS

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The stability of periodic solutions of quasilinear elastic slightly asymmetric gyroscopic systems with distributed and concentrated parameters is considered. The motion investigated is described by a system of partial differential equations; the boundary conditions and matching conditions at the sites of the concentrated parameters also take the form of quasilinear equations. The nonlinear functions in the equations of motion and in the boundary conditions are assumed to be of sufficiently general form; this makes it possible to investigate the stability of the periodic solutions under the most varied perturbations. It is assumed that some of the natural frequencies of the linearized system can be critical or resonance frequencies. The gyroscopic effect of the distributed mass is assumed to be negligibly small, as usual.

The periodic oscillation states of unbalanced flexible rotors, some of whose supports have nonlinear characteristics, are constructed as an example. The equations in variations are written out and it is shown that their stability can be investigated completely by the proposed method.

1. Many problems of applied mechanics involve the action on quasilinear elastic gyroscopic systems of periodic forces whose frequencies are usually multiples of the angular velocity ω of the gyro system rotor. Their equations of motion have periodic solutions with the period $T = 2\pi / \omega$; however, these solutions may turn out to be unstable for various reasons, so that almost-periodic autooscillatory states (not always permissible ones) arise in the gyro system. The conditions of stability of the periodic oscillations therefore assume considerable importance.

The motion of elastic gyro systems is described in the more complicted cases by a